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Statistical Entropy of Reissner-Nordstrom Black Hole Computed by Generalized Uncertainty Principle

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Abstract The equation of state density is corrected by the generalized uncertainty principle. Statistical entropy of scalar field outside Reissner-Nordstrom black hole is computed by WKB approximation method. The result shows that this black hole entropy is proportional to its horizon area, which is the same as that given by brick-wall method. The difference from the brick-wall method is that the present result is convergent without any cutoff.

Keywords Generalized uncertainty principle · Reissner-Nordstrom black hole · State density · Statistical entropy

In recent 30 years, research on black hole has made considerable progress. The black hole theory collecting general theory of relativity, quantum theory and statistical thermodynamics into a body has become a hot spot in the research of current physics and astronomy. But people's understanding to the black hole entropy is actually not so satisfactory. Since the 1970s, Bekenstein and Hawking have proved that the entropy of black holes is directly proportional to the event horizon area [1, 2], and the statistical origin of black hole entropy has become an important subject in the research of theoretical physics. Because of the statistical significance of black holes entropy, the understanding to entropy involves its microscopic essence. Maybe a complete quantum gravitational theory is needed for us to truly understand microscopic origin of the black hole entropy. In present situation, quantum gravitational theory is not advanced enough and people develop a variety of semi-classic methods to explore microscopic origin of the black hole entropy. The common methods include brick-wall method, conical singularity, blunt-cone and volume cut-off, in which brick-wall method [3] proposed by 't Hooft in 1985 is most widely used. Although the conclusion that the entropy is proportional to the event horizon area has been obtained by brick-wall method, a cut-off has to be introduced to avoid divergence, which looks unnatural. Is there any other way that can be used to remove the divergence without any cut-off.

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Recently Chang points out that, if considering the correction to uncertainty relation in curved space, the equation of state density will be modified at the same time [4]. Li Xiang proposes that Chang's conclusion can be applied for solving the divergence problem [5]. At present people believe gravitation effect will cause the existence of Plank length, the shortest distance in which space-time is discontinuous. And the generalized uncertainty relation is brought about in such background [6].

1 Generalized Position-Momentum Uncertainty Relation

Position-momentum uncertainty relation in Minkowski space-time is given by

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{1}$$

Chang points out that, if we consider the gravitation, the position-momentum uncertainty relation is expanded as follows [4]

$$\Delta x \Delta p \geq \frac{1}{2}\left[\hbar + \frac{\lambda}{\hbar}(\Delta p)^2\right], \tag{2}$$

where $\hbar$ is the Planck constant, $\sqrt{\lambda}$ is of order of the Planck length. It can be obtained easily from (1) that

$$\Delta x \geq \frac{1}{2}\left[\frac{\hbar}{\Delta p} + \frac{\lambda}{\hbar}\Delta p\right] \geq \sqrt{\frac{\hbar}{\Delta p} \cdot \frac{\lambda}{\hbar}\Delta p} = \sqrt{\lambda}. \tag{3}$$

This means that there is a minimal length $\sqrt{\lambda}$. In statistical physics, one quantum state corresponds to a "cell" of volume $(2\pi\hbar)^3$ in phase-space based on the uncertainty relation (1). So the number of quantum states for $dx^3 dp^3$ is given by

$$\frac{dx^3 dp^3}{(2\pi\hbar)^3}. \tag{4}$$

Correspondingly, when the generalized uncertainty relation namely (2) is considered, the number of quantum states should be changed to [4]

$$\frac{dx^3 dp^3}{(2\pi\hbar)^3(1+\lambda p^2)^3}, \tag{5}$$

where $p^2 = p_i p^i$ $(i = 1, 2, 3)$. In this paper, we will adopt the modified state density equation (5) to calculate the statistical entropy of Reissner-Nordstrom black hole.

2 Klein-Gordon Equation in Reissner-Nordstrom Space-Time

Using Natural units ($G = c = \hbar = 1$), the metric of Reissner-Nordstrom black hole is

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -\left(1 - \frac{2mr - Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \end{aligned} \tag{6}$$

where m is the mass of black hole, Q is the electric charge of black hole. So we can obtain

$$
\begin{aligned}
&g_{00} = -\left(1 - \frac{2mr - Q^2}{r^2}\right), \qquad g_{11} = \left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1}, \\
&g_{22} = r^2, \qquad g_{33} = r^2 \sin^2\theta, \\
&g = |g_{\mu\nu}| = -r^4 \sin^2\theta, \\
&g^{00} = -\left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1}, \qquad g^{11} = \left(1 - \frac{2mr - Q^2}{r^2}\right), \\
&g^{22} = r^{-2}, \qquad g^{33} = r^{-2}\sin^{-2}\theta.
\end{aligned}
\tag{7}
$$

Klein-Gordon equation of a free scalar particle in Reissner-Nordstrom space-time is

$$
\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}\,g^{\mu\nu}\frac{\partial\Phi}{\partial x^\nu}\right] = 0, \tag{8}
$$

where x^0, x^1, x^2, x^3 correspond to t, r, θ, φ separately. Putting the values of (7) into (8), we get

$$
\begin{aligned}
&-\left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1}\frac{\partial^2\Phi}{\partial t^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left[(r^2\sin\theta - 2m\sin\theta r + Q^2\sin\theta)\frac{\partial\Phi}{\partial r}\right] \\
&\quad + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0.
\end{aligned}
\tag{9}
$$

To separate the variable t, we substitute $\Phi = e^{-i\omega t}\Psi(r, \theta, \varphi)$ into (9) and obtain

$$
\begin{aligned}
&\left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1}\omega^2\Psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left[(r^2\sin\theta - 2m\sin\theta r + Q^2\sin\theta)\frac{\partial\Psi}{\partial r}\right] \\
&\quad + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} = 0.
\end{aligned}
\tag{10}
$$

3 Computation of the Modified State Density and Statistical Entropy

Using the WKB approximation with $\Psi = \exp[is(r, \theta, \varphi)]$, we turn (10) into

$$
\begin{aligned}
&\left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1}\omega^2 e^{is} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left[(r^2\sin\theta - 2m\sin\theta r + Q^2\sin\theta)e^{is}i\frac{\partial s}{\partial r}\right] \\
&\quad + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta e^{is}i\frac{\partial s}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left[-e^{is}\left(\frac{\partial s}{\partial\varphi}\right)^2 + e^{is}i\frac{\partial^2 s}{\partial\varphi^2}\right] = 0.
\end{aligned}
$$

We only need the real part of the above equation and get

$$
\begin{aligned}
&\left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1}\omega^2 + \frac{1}{r^2\sin\theta}(r^2\sin\theta - 2m\sin\theta r + Q^2\sin\theta)p_r^2 \\
&\quad - \frac{1}{r^2}p_\theta^2 - \frac{1}{r^2\sin^2\theta}p_\varphi^2 = 0,
\end{aligned}
\tag{11}
$$

where $p_r = \frac{\partial s}{\partial r}$, $p_\theta = \frac{\partial s}{\partial \theta}$, $p_\varphi = \frac{\partial s}{\partial \varphi}$. From (11) we obtain

$$p_r^2 = \left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1}\left[\left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1}\omega^2 - \frac{1}{r^2}p_\theta^2 - \frac{1}{r^2\sin^2\theta}p_\varphi^2\right]. \tag{12}$$

From the above (12) we obtain

$$\begin{aligned} p^2 &= p_i p^i \\ &= g^{\mu\nu} p_i p_i \\ &= g^{rr} p_r^2 + g^{\theta\theta} p_\theta^2 + g^{\varphi\varphi} p_\varphi^2 \\ &= \left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1}\omega^2. \end{aligned} \tag{13}$$

From (5) and (12), the number of quantum states with energy less than ω outside Reissner-Nordstrom black hole is given by

$$\begin{aligned} g(\omega) &= \frac{1}{(2\pi)^3}\iiint\iiint \frac{dr d\theta d\varphi dp_r dp_\theta dp_\varphi}{(1+\lambda p^2)^3} \\ &= \frac{1}{(2\pi)^3}\iiint \frac{dr d\theta d\varphi}{(1+\lambda p^2)^3}\iint dp_\theta dp_\varphi \int_{-p_r}^{+p_r} dp_r \\ &= \frac{1}{(2\pi)^3}\iiint \frac{dr d\theta d\varphi}{(1+\lambda p^2)^3}\iint \frac{2}{\sqrt{1 - \frac{2mr-Q^2}{r^2}}}\sqrt{\frac{\omega^2}{1-\frac{2mr-Q^2}{r^2}} - \frac{1}{r^2}p_\theta^2 - \frac{p_\varphi^2}{r^2\sin^2\theta}}dp_\theta dp_\varphi \\ &= \frac{1}{6\pi^2}\int \frac{r^2\omega^3 dr}{(1+\lambda p^2)^3(1 - \frac{2mr-Q^2}{r^2})^2}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \\ &= \frac{2}{3\pi}\int \frac{r^2\omega^3}{(1+\lambda p^2)^3(1 - \frac{2mr-Q^2}{r^2})^2}dr, \end{aligned} \tag{14}$$

where the integration goes over those values of p_θ, p_φ for which the argument of the square root is positive. The $g(\omega)$ of (14) is convergent at the horizon, so there is no need to introduce any cutoff. The free energy of scalar field in Reissner-Nordstrom space-time is given by

$$\begin{aligned} F(\beta) &= \frac{1}{\beta}\int dg(\omega)\ln(1 - e^{-\beta\omega}) \\ &= \frac{1}{\beta}\left[\ln(1 - e^{-\beta\omega})g(\omega)|_0^\infty - \int g(\omega)d\ln(1 - e^{-\beta\omega})\right] \\ &= \frac{1}{\beta}\left[0 - \int_0^\infty \frac{g(\omega)}{1 - e^{-\beta\omega}}d(1 - e^{-\beta\omega})\right] \\ &= -\int_0^\infty \frac{g(\omega)}{e^{\beta\omega} - 1}d\omega \\ &= -\frac{2}{3\pi}\int \frac{r^2 dr}{(1 - \frac{2mr-Q^2}{r^2})^2}\int_0^\infty \frac{\omega^3 d\omega}{(1+\lambda p^2)^3(e^{\beta\omega} - 1)}. \end{aligned} \tag{15}$$

The statistical entropy of scalar field in Reissner-Nordstrom space-time reads

$$\begin{aligned} S &= \beta^2 \frac{\partial F}{\partial \beta} \\ &= \frac{2\beta^2}{3\pi} \int \frac{r^2 dr}{(1-\frac{2mr-Q^2}{r^2})^2} \int_0^\infty \frac{e^{\beta\omega}\omega^4 d\omega}{(e^{\beta\omega}-1)^2(1+\lambda p^2)^3}. \end{aligned}$$

Using (13) and making $x=\beta\omega$ for the above equation, we get

$$\begin{aligned} S &= \frac{2\beta^2}{3\pi} \int \frac{r^2 dr}{(1-\frac{2mr-Q^2}{r^2})^2} \int_0^\infty \frac{e^x (\frac{x}{\beta})^4 \frac{dx}{\beta}}{(e^x-1)^2[1+\lambda x^2/\beta^2(1-\frac{2mr-Q^2}{r^2})]^3} \\ &= \frac{2\beta^{-3}}{3\pi} \int \frac{r^2 dr}{(1-\frac{2mr-Q^2}{r^2})^2} \int_0^\infty \frac{x^4 dx}{(e^x-1)(1-e^{-x})[1+\lambda x^2/\beta^2(1-\frac{2mr-Q^2}{r^2})]^3}. \end{aligned} \tag{16}$$

Using the following inequality

$$1-e^{-x} > \frac{x}{1+x}, \qquad e^x - 1 > x, \tag{17}$$

we obtain

$$\begin{aligned} S &< \frac{2\beta^{-3}}{3\pi} \int \frac{r^2 dr}{(1-\frac{2mr-Q^2}{r^2})^2} \int_0^\infty \frac{(x^3+x^2)dx}{[1+\lambda x^2/\beta^2(1-\frac{2mr-Q^2}{r^2})]^3} \\ &= \frac{\beta}{6\pi\lambda^2} \int r^2 dr + \frac{1}{24\lambda^{\frac{3}{2}}} \int \frac{r^2}{\sqrt{1-\frac{2mr-Q^2}{r^2}}} dr. \end{aligned} \tag{18}$$

We shall calculate the entropy only in the vicinity near the horizon since the entropy of the black hole is mainly attributed from the quantum field near the horizon. From (3) we know $\sqrt{\lambda}$ is the minimal length, so we only analyses the thin layer with a proper distance of $\sqrt{\lambda}$ in the radial range of $[r_+, r_+ + \varepsilon]$. To find the relation between ε and λ, g_{00} is expanded with Taylor series at $r = r_+$ as follows

$$g_{00} \approx g_{00}(r_+) + g'_{00}(r_+)(r - r_+). \tag{19}$$

The surface gravity of Reissner-Nordstrom black hole is [7]

$$\kappa = -\frac{1}{2} \lim_{r\to r_+} \sqrt{\frac{-g^{rr}}{g^{00}}} \frac{\partial}{\partial r} \ln(-g^{00}). \tag{20}$$

With (7), (21) is simplified to

$$\kappa = -\frac{1}{2} \lim_{r\to r_+} \frac{\partial}{\partial r}(g_{00}). \tag{21}$$

Substituting $g_{00}(r_+) = 0$ and (21) into (19) yields

$$g_{00} \approx -2\kappa(r - r_+). \tag{22}$$

Therefore we get the relation between ε and λ as follows

$$\begin{aligned}\sqrt{\lambda} &= \int_{r_+}^{r_+ +\varepsilon} \sqrt{g_{rr}}dr \\ &= \int_{r_+}^{r_+ +\varepsilon} \sqrt{\frac{1}{-g_{00}}}dr \\ &\approx \int_{r_+}^{r_+ +\varepsilon} \frac{dr}{\sqrt{2\kappa(r-r_+)}} \\ &= \sqrt{\frac{2\varepsilon}{\kappa}}.\end{aligned} \tag{23}$$

On the horizon surface $r = r_+$, t is constant. So we have

$$\begin{aligned}ds^2 &= r_+^2(d\theta^2 + \sin^2\theta d\varphi^2), \\ g &= r_+^4 \sin^2\theta, \\ A &= \iint \sqrt{g}d\theta d\varphi = 4\pi r_+^2,\end{aligned} \tag{24}$$

where A is the event horizon area of the black hole. From (18) and (23), we obtain

$$\begin{aligned}S &< \frac{\beta}{6\pi\lambda^2}r_+^2\varepsilon + \frac{1}{24\lambda^{\frac{3}{2}}}\int_{r_+}^{r_+ +\varepsilon} r^2\sqrt{g_{rr}}dr \\ &< \frac{\beta}{6\pi\lambda^2}r_+^2\varepsilon + \frac{1}{24\lambda^{\frac{3}{2}}}r_+^2\sqrt{\lambda}.\end{aligned} \tag{25}$$

In black hole thermodynamics, we have [7]

$$\beta = \frac{2\pi}{\kappa}.$$

So (25) is simplified as

$$S \approx \frac{5A}{96\pi\lambda}. \tag{26}$$

Equation (26) indicates that the statistical mechanics entropy of Reissner-Nordstrom black hole is directly proportional to its event horizon area, which is the same as other results deduced in other ways. When using the general uncertainty relation, we need not to introduce any cutoff, which implies the connection between black hole entropy and quantum effect near the horizon.

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